I-5. RAY THEORY OF RESONATORS AND BEAM WAVEGUIDES WITH AN INHOMOGENEOUS MEDIUM

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Important characteristics of optical resonators and beam waveguides can be inferred from simple ray-optical considerations. The ray theory is susceptible to an extremely convenient algebraic formulation from which a formal analogy in terms of electrical networks is evident. Most of the available results for optical resonators and beam waveguides assume the medium within the structure to be homogeneous and isotropic. In laser applications of optical resonators the medium is certainly inhomogeneous and, furthermore, dispersive. A ray analysis is carried through for these complex cases, and yields some very interesting and general conclusions. In particular, the stability of the ray systems, i.e., discrimination between high and low-loss configurations, is discussed in terms of a generalization of Pierce's criterion. For simplicity, two-dimensional systems are considered; the extension to three dimensions follows directly.

The close connection between rays within an optical resonator and rays on an (infinite) beam waveguide is illustrated in Figure 1. The focal lengths of the curved mirror comprising the optical cavity are equal to the focal lengths of the simple lenses of the beam waveguide. Now consider the development of initially parallel and similarly placed rays within each structure. Ray segments within the resonator which have undergone an even number of reflections are parallel to ray segments on the beam waveguide, while ray segments which have undergone an odd number of reflections within the resonator are mirror images of the corresponding ray segments on the beam waveguide.

If we label a series of planes transverse to the axis of our optical system successively ... , N - 1, N, N + 1, ..., then at each of these planes a ray is characterized by two parameters, e.g., the distance from the axis a and the slope of the ray b. These two parameters may be sub-sumed in a column matrix. For successive reference planes, these columns are then related by a transformation characteristic of the intervening optical system. An illustration is provided by Figure 2, where the Nth and (N + 1)th plane are separated only by a distance ℓ. Clearly

$$\begin{pmatrix} a_{N+1} \\ b_{N+1} \end{pmatrix} = \begin{pmatrix} 1 & \ell' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_N \\ b_N \end{pmatrix} \qquad (1)$$

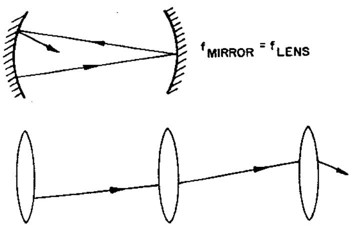

Figure 1. An Optical Resonator and Equivalent Beam Waveguide

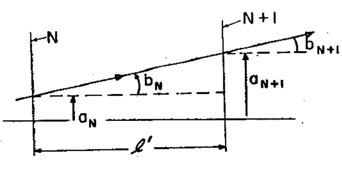

Figure 2. Algebraization of Rays

For transverse planes separated only by a simple lens (of zero thickness) the transformation matrix is

$$\begin{pmatrix} 1 & 0 \\ -c' & 1 \end{pmatrix} \tag{2}$$

where c', the reciprocal focal length, is considered positive for a convergent lens.

The behavior of any resonator, for example, the simple one shown in Figure 1, may be discussed in terms of a unit-cell of the beam waveguide. A convenient symmetric unit-cell is indicated in Figure 3. The transformation matrix T for this unit cell is simply the product of the transformation matrices for each of the constituents:

$$\begin{aligned} T &= \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2c & 1 \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & Z\sin\theta \\ -Y\sin\theta & \cos\theta \end{pmatrix} \end{aligned} \tag{3}$$

where

$$\cos\theta = 1 - 2\ell c \quad \text{and} \quad Z = 1/Y = \ell \cot(\theta/2) .$$

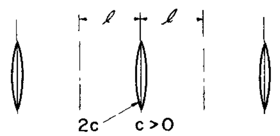

Figure 3. Symmetric Unit Cell of Simple Beam Waveguide

The form, Equation (3), is particularly convenient since from it T^N is obtained simply by replacing θ by $N\theta$.

The forms of Equations (1) through (3) provide a convenient basis for development of the formal analogy with electrical networks also remarked by Deschamps and Mast.

One gross physical indicator of the electromagnetic state of affairs which obtains on either of these two structures is the characteristic behavior (paraxial) rays. Such rays either remain closely confined within the structure or they diverge (to infinite distances) from the structure. A criterion for distinguishing between these two situations was developed by Pierce and more recently applied by Boyd and Kogelnik to delimit high-loss and low-loss configurations among optical resonators formed by two mirrors of different curvatures. As these limits have proved of considerable importance to experimenters, a convenient, generalized formulation of Pierce's criterion was developed.

It may be shown quite generally that in terms of the transformation matrix for a unit cell, Pierce's criterion takes the simple form

$$-1 < \frac{1}{2} \text{trace of } T < 1 . \tag{4}$$

In particular, for the elementary case represented by Figure 3 and Equation (3), one recovers the well-known result (in our notation)

$$0 < \ell c < 1 . \tag{5}$$

Again, it may be shown quite generally that, for any ray, the quantity

$$a_N^2 \pm |Z|^2 b_N^2 = \text{constant} \tag{6}$$

independent of N. The upper sign applies for stable configurations; the lower sign applies for unstable configurations. Equation (6) has an interesting geometrical interpretation. Consider the development of the ray

$$\begin{pmatrix} a_m \\ 0 \end{pmatrix} \tag{7}$$

at successive reference planes within the resonator or beam waveguide. If the configuration is stable the hyperbola shown in Figure 4(a) forms the envelope of the ray system constructed by drawing lines with the required position and slope at the reference planes. If the configuration is unstable, a similar role is played by the ellipse shown in Figure 4(b). When the resonator or guide is immersed in a homogeneous medium (and only then), the rays so constructed actually correspond to physical rays.

Most of the results on optical resonators obtained previously disregard possibly special properties of the medium which fills the cavity; i.e., it is assumed that the medium is homogeneous and isotropic. In laser applications, the medium is probably inhomogeneous and, furthermore, is dispersive. As already indicated, it is quite feasible to take account of the possibility that the medium within the resonator is inhomogeneous. From the optical standpoint, two elementary cases come to mind: the medium is either convergent or divergent. Let z be a measure distance along the axis of the resonator and y a coordinate transverse to the axis; then the two cases correspond to variations of the index of refraction n

$$n(y) = \eta_a + \eta y^2, \quad \left|\frac{\eta}{\eta_a}\right| << 1, \tag{8}$$

near the axis of the resonator, and where for a convergent medium $\eta < 0$ and a divergent medium $\eta > 0$. The differential equation for the rays in such media reduces in the paraxial limit to

$$\frac{d^2}{dz^2} y = \frac{2\eta}{\eta_a} y. \tag{9}$$

Thus, for a convergent medium, the paraxial rays are easily expressed in terms of trigonometric functions of the coordinate z (hyperbolic functions for divergent medium). The transformation matrix for a length z of the medium is

$$\begin{pmatrix} \cos \gamma z & Z \sin \gamma z \\ -Y \sin \gamma z & \cos \gamma z \end{pmatrix} \tag{10}$$

where

$$\gamma = \sqrt{\frac{2\eta}{\eta_a}} = Y. \tag{11}$$

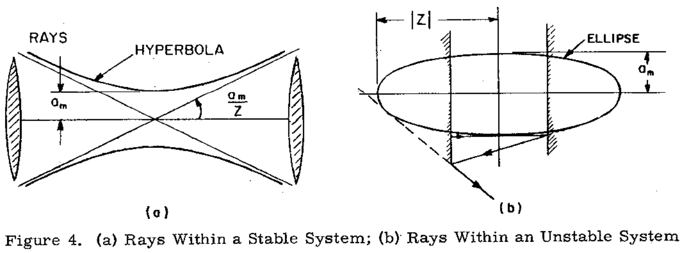

Figure 4. (a) Rays Within a Stable System; (b) Rays Within an Unstable System

It is evident on comparison of Equations (10) and (3) that the inhomogeneous medium in fact acts as a beam waveguide. In fact, in the limit

$$\ell << 1, \quad c << 1 , \tag{12}$$

one obtains the direct correspondence

$$\frac{2\eta}{\eta_a} = \frac{c}{\ell} . \tag{13}$$

Now consider a resonator or beam waveguide filled with an inhomogeneous medium. A resonator filled with a convergent medium is shown in Figure 5. The transformation matrix for a unit cell is

$$\begin{pmatrix} \cos 2\gamma\ell - KZ \sin 2\gamma\ell & Z \sin 2\gamma\ell \\ -2K \cos 2\gamma\ell + (K^2Z - Y) \sin 2\gamma\ell & \cos 2\ell - KZ \sin 2\gamma\ell \end{pmatrix} \tag{14}$$

wherein γ and Z are given by Equation (11). From this matrix one finds that Pierce's criterion is

$$-1 < \cos 2\gamma\ell - KZ \sin 2\gamma\ell < +1 , \tag{15}$$

or, defining $\tan \phi = KZ$,

$$-1 < \sec \phi \cos (2\gamma\ell + \phi) < +1 .$$

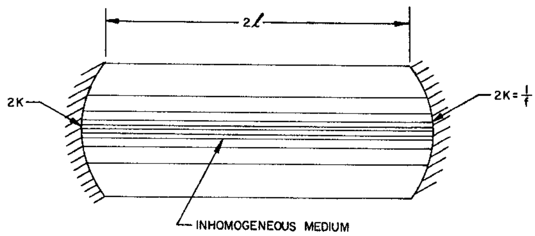

Figure 5. Resonator with Convergent Inhomogeneous Medium

Thus it is seen that for a resonator with a convergent medium, stable and unstable regions recur periodically for arbitrary separations of the resonator plates, 2ℓ.

The characteristic behavior of three cases: resonator empty, resonator filled with convergent medium, resonator filled with divergent medium, is summarized in Figure 6.

In the neighborhood of an amplifying transition, a maser medium is also dispersive. When this dispersive character is coupled with effects due to inhomogeneities of the types discussed, gross dynamic changes in the effective Q of optical resonators may be anticipated.

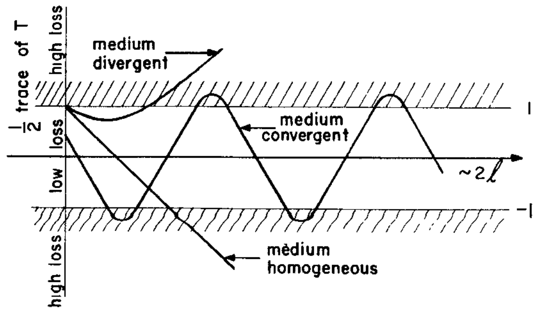

Figure 6. Characteristic Stability Diagram for Optical Resonators and Beam Waveguides

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